

# Generating resonating valence bond states through Dicke subradiance

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Dicke's original thought experiment with two spins coupled to a photon mode has recently been experimentally realized. We propose extending this experiment to  $N$  spins and show that it naturally gives rise to highly entangled states. In particular, it gives rise to dark states which have resonating valence bond (RVB) character. We first consider a system of  $N$  two-level spins in a cavity with only one spin in the excited state. This initial state is a linear combination of a dark state with total spin  $S = N/2 - 1$  and a bright state with total spin  $S = N/2$ . We point out the dark state is a coherent superposition of singlets with resonating valence bond character. We show that the coupling to the photon mode takes the spin system into a mixed state with an entangled density matrix. We next consider an initial state with half of the spins in the excited state. We show that there is a non-zero probability for this to collapse into a dark state with RVB character. In the lossy cavity limit, if no photon is detected within several decay time periods, we may deduce that the spin system has collapsed onto the dark RVB state. We show that the probability for this scales as  $N^{-1}$ , making it possible to generate RVB states of 20 spins or more.

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*Introduction:* Dicke, in his seminal 1954 paper[1], showed that two-level systems coupled to a common photon mode will decay coherently. He begins his argument with a classic thought experiment considering two two-level systems – one in the excited state and the other in the ground state. Naïve intuition suggests that this system will emit a single photon. However, upon mapping the two-level systems to spin-1/2 spins, the initial state is a linear combination of a dark singlet state and a bright triplet state. Hence, a photon will only be emitted with probability half. Building on this two spin experiment, Dicke argued that outgoing radiation from an  $N$ -spin system will be dominated by coherent photons emitted by ‘superradiant’ states which have maximal  $S$ , the total spin angular momentum.

The notion of superradiance has been realized and tested in many physical contexts, e.g., Bose-Einstein condensates[2], nuclear spins[3], magnons[4], excitons in quantum dots[5], etc. Recently, Mlynek et al have recreated Dicke's original thought experiment in the lab using two superconducting qubits in a microwave cavity[6]. They succeeded in measuring the density matrix of the emitted photon, confirming Dicke's prediction. Their result may be expressed in the following way. Two otherwise isolated spins were entangled by their coupling to a common photon field; the entanglement was quantified by measuring the photon output. Expressed this way, this experiment heralds a new method to create entangled quantum states in the lab.

Generating entangled states in the laboratory has been a long standing goal. There are well established ways to generate two entangled photons using polarized photons[7]. However, it remains a challenge to generate multi-particle states with entanglement despite several

recent successes[8–10]. Several studies have examined whether Dicke superradiance could be used to generate entanglement[11–13]. In contrast, several recent studies have focussed on ‘subradiance’ as a means for storing information in long-lived states[14, 15].

In this letter, we point out that subradiant states of Dicke systems are highly entangled. Moreover, they form a particularly rich class of entangled states called resonating valence bond (RVB) states. RVB states are many body singlets first proposed by Pauling in the context of Benzene. RVB states have been proposed to be precursors of high temperature superconductivity[16] and are known to have topological order[17]. RVB states on four-site plaquettes have been experimentally realized using ultracold atoms[18, 19]. We suggest two experiments to generate entangled RVB states of  $N$  spins. In the first, we consider an initial state with a single excitation and show how the outgoing photon entangles the  $N$  spins. In the second, we take half the spins to be in the excited state initially and suggest a protocol to isolate the dark component, which is precisely an RVB state.

*Dicke's thought experiment:* Consider two qubits with one in the excited state and the other in the ground state. The qubits are assumed to be in close proximity within a cavity resonator, so that they couple to a photon mode with the same coupling strength. The distance between them,  $r$ , is much smaller than the photon wavelength ( $r \ll \lambda$ ) so that their coupling constants do not incur a phase difference. At the same time, the distance is large when compared to the particle wavelength so that the particles' wavefunctions do not overlap and the inter-qubit interaction can be neglected.

This initial state is a superposition of the singlet state  $|s\rangle = \{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\}/\sqrt{2}$  and a triplet state  $|t_0\rangle = \{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$

$\rangle\rangle/\sqrt{2}$ . When interacting with the radiation field, the triplet part emits a photon and jumps to the  $|t_{-1}\rangle$  triplet state. The singlet state remains a spectator, coupling neither to the photon nor to the triplet states. If no photon is emitted for a sufficiently long period of time, we may surmise that the two qubits are in the singlet state. It is then impossible to say which qubit is in the excited state.

*Extension to  $N$  spins:* Following Dicke's arguments, the  $N$ -qubit system may be thought as a wavefunction for  $N$  spin-1/2 moments, expressed in terms of the total angular momentum states  $|S_{tot}, m_{tot}\rangle$ [20]. The spin operators for total spin are given by the sum of the individual qubit operators  $\hat{S}_{tot}^\alpha = \hat{S}_1^\alpha + \dots + \hat{S}_N^\alpha$ , with  $\alpha = x, y, z$ . Within the rotating wave approximation[21], the Dicke Hamiltonian for the system of spins coupled to the photon mode can be written as

$$H = \lambda \hat{S}_{tot}^z + \omega_c a^\dagger a + g\{\hat{S}_{tot}^- a^\dagger + \hat{S}_{tot}^+ a\}. \quad (1)$$

The first two terms represent the energy of the spins and the photon mode respectively. The third term denotes the coupling between spins and the photons. We assume that the energy splitting of the spin states is close to one of the resonance frequencies of the cavity, so that we may neglect all other modes.

*Initial state with single excitation:* We take the simplest non-trivial initial state,  $|\uparrow\downarrow \dots \downarrow\rangle$ , i.e., only one qubit is in the excited state while the other  $N-1$  qubits are in the ground state. To understand photon emission from this state, it is convenient to reexpress this state using eigenstates of the total angular momentum operators. As this state has the spin- $z$  component  $m_{tot} = -N/2 + 1$ , its projection on  $|S_{tot}, m_{tot}\rangle$  can only be non-zero when  $S_{tot} = N/2$  or  $S_{tot} = N/2 - 1$ . For any other value of  $S_{tot}$ , we cannot have  $m_{tot} = -N/2 + 1$  as  $m_{tot}$  is constrained to take values  $-S_{tot}, \dots, S_{tot}$ . In analogy with the two-spin problem, the  $S_{tot} = N/2$  component is bright; it will emit a photon and decay to  $|S_{tot} = N/2, m_{tot} = -N/2\rangle$ . The  $S_{tot} = N/2 - 1$  component is dark as it cannot reduce its  $m_{tot}$  quantum number further.

The bright component with  $S_{tot} = N/2$  is superradiant, i.e., it has maximal  $S$  for the given number of spins. Superradiant states are symmetric under permutation of spins, which is a symmetry of the Dicke Hamiltonian itself. The projector operator  $\hat{\mathcal{P}}_{S=N/2}$  is therefore simply the symmetrization operator[22]. The bright component in the initial state is given by

$$\mathcal{N}\{\hat{\mathcal{P}}_{S=N/2}|\uparrow\downarrow \dots \downarrow\rangle\} = \frac{1}{\sqrt{N}}\left[|\uparrow\downarrow \dots \downarrow\rangle + \dots + |\downarrow \dots \uparrow\downarrow\rangle\right], \quad (2)$$

where  $\mathcal{N}\{\}$  denotes the normalization operation. The amplitude of the bright component thus decays as  $1/\sqrt{N}$ . The dark component with  $S_{tot} = N/2 - 1$  is simply obtained by subtracting out the superradiant component

FIG. 1: Dark component of  $N$  spin state with one spin up, with  $N = 4, 5$ . The dark state is obtained by action of the projection operator  $\hat{\mathcal{P}}_{S=N/2-1}$ , followed by normalization. The result is a linear combination of single-dimer states. The normalization constant for general  $N$  is  $\sqrt{2/(N(N-1))}$ .

and normalizing the result,

$$\mathcal{N}\{\hat{\mathcal{P}}_{S=N/2-1}|\uparrow\downarrow \dots \downarrow\rangle\} = \sqrt{\frac{N-1}{N}}|\uparrow\downarrow \dots \downarrow\rangle - \frac{1}{\sqrt{N(N-1)}}\left[|\downarrow\uparrow\downarrow \dots \downarrow\rangle + |\downarrow\downarrow\uparrow\downarrow \dots \downarrow\rangle + \dots\right]. \quad (3)$$

The amplitude of the dark component is given by  $\sqrt{1-1/N}$ . The probability of photon emission falls as  $1/N$  and vanishes in the large- $N$  limit. This is the well known phenomenon of radiation trapping[23, 24] which can be interpreted as a photon being continually absorbed and reemitted among  $N$  spins. Equivalently, the different decay pathways interfere destructively[25].

Remarkably, the dark state in Eq. 3 has resonating valence bond character. The spin which was initially in the excited state forms a singlet with one of the other  $N-1$  spins. This singlet bond ‘resonates’ leading to a symmetric linear combination of  $N-1$  states, each with one dimer. This is depicted in Fig. 1 for  $N = 4, 5$ .

As demonstrated in Refs. 26 and 6, the emitted photon can be detected and its density matrix constructed. In our theoretical analysis, the wavefunction of the combined spin-photon system is

$$|\Psi_F\rangle = \sqrt{\frac{N-1}{N}}|S_{tot} = N/2 - 1, m_{tot} = -N/2 + 1\rangle \otimes |0\rangle + \frac{1}{\sqrt{N}}|S_{tot} = N/2, m_{tot} = -N/2\rangle \otimes |1\rangle. \quad (4)$$

The expected reduced density matrix for the photon is

$$\hat{\rho}_{ph} = \text{Tr}_{spin}|\Psi_F\rangle\langle\Psi_F| = \left[\frac{1}{N}|1\rangle\langle 1| + \frac{(N-1)}{N}|0\rangle\langle 0|\right]. \quad (5)$$

We may equally well trace out the photons and obtain a mixed density matrix for the spins in the  $|S_{tot}, m_{tot}\rangle$  basis. In either case, the reduced density matrix reveals a mixed state, with the von Neumann entanglement entropy given by

$$S_{ph/spin} = -\text{Tr}\rho_{ph} \ln \rho_{ph} = \ln(N) - \frac{N-1}{N} \ln(N-1). \quad (6)$$

$S_{ph/spin}$  is a slowly decaying function of  $N$ , showing significant entanglement even up to  $N \sim 100$  spins.



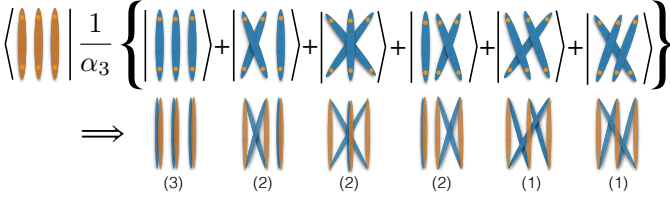


FIG. 3: Evaluating overlap of one particular dimer cover with the RVB state for  $N = 6$  spins. The resulting overlap graphs are shown below each ket, with the number of loops ( $N_l$ ) in parantheses.

$|\Psi_1\rangle$  with the RVB state as shown in Fig. 3. Using Eq. 8, the overlap comes out to be

$$\langle \Psi_1 | \Psi_{singlet} \rangle = \langle \Psi_1 | \left[ \frac{1}{\alpha_D} \sum_P |\Psi_P\rangle \right] = \frac{1}{\alpha_D 2^D} \sum_P 2^{N_l(P)}. \quad (9)$$

The index  $P$  sums over permutations of the spins in the bottom row – each permutation corresponds to one component of the RVB state.  $N_l(P)$  is the number of loops obtained from the overlap graph for the given permutation  $P$ . By symmetry, this overlap is the same for any choice of the dimer cover state  $|\Psi_1\rangle$ . Consequently, the overlap of the RVB state with itself is

$$\begin{aligned} \langle \Psi_{singlet} | \Psi_{singlet} \rangle &= \left[ \frac{1}{\alpha_D} \sum_P \langle \Psi_P | \right] |\Psi_{singlet}\rangle \\ &= \frac{(D)!}{\alpha_D^2 2^D} \sum_P 2^{N_l(P)}. \end{aligned} \quad (10)$$

We have used the fact that the RVB state is a symmetric linear combination of  $D!$  dimer covers. The quantity  $\sum_P 2^{N_l(P)}$  can be evaluated using Burnside's lemma, a well known result in combinatorics. Here, we apply Burnside's lemma to the group of permutations of  $D$  objects to obtain (see Appendix for details)

$$\sum_P 2^{N_l(P)} = (D+1)D!. \quad (11)$$

Demanding that the RVB state be normalized, we obtain

$$\alpha_D = D! \sqrt{\frac{(D+1)}{2^D}}. \quad (12)$$

*Amplitude of RVB component in initial state:* We have determined that  $|\Psi_{singlet}\rangle = \mathcal{N} \left\{ \hat{P}_0 |\Psi_{init}\rangle \right\}$  has RVB character. We have determined its normalization. To evaluate the probability that the initial state will collapse onto the RVB state, we first evaluate the overlap of a single dimer cover with  $|\Psi_{init}\rangle$ , the initial state. We have

$$\langle \Psi_1 | \Psi_{init} \rangle = \left( \frac{1}{\sqrt{2}} \right)^D, \quad (13)$$

as the overlap decomposes into overlaps over each dimerized pair. Each dimer (singlet wavefunction) in  $|\Psi_1\rangle$  is overlapped with  $|\uparrow\downarrow\rangle$  from  $|\Psi_{init}\rangle$ , contributing a factor of  $1/\sqrt{2}$ . By symmetry, the overlap is the same for any choice of dimer cover  $|\Psi_1\rangle$ . We have

$$\langle \Psi_{singlet} | \Psi_{init} \rangle = \frac{1}{\alpha_D} \sum_P \langle \Psi_P | \Psi_{init} \rangle, \quad (14)$$

where  $P$  sums over all dimer covers (equivalent to permutations of  $D$  objects). As there are  $D!$  possible dimer covers, we obtain

$$\langle \Psi_{singlet} | \Psi_{init} \rangle = \frac{D!}{\alpha_D} \left( \frac{1}{\sqrt{2}} \right)^D. \quad (15)$$

It follows that the probability of the initial state collapsing onto the RVB singlet is given by

$$|\langle \Psi_{singlet} | \Psi_{init} \rangle|^2 = \frac{1}{D+1} = \frac{2}{N+2}. \quad (16)$$

This is the key result of this letter: the probability of the initial state collapsing into a dark RVB singlet decays very slowly with  $N$  paving the way for experimental realization with reasonably large values of  $N$ .

*Discussion* With appropriate initial states of  $N$  spins in a cavity, a null measurement for photon emission collapses the state onto a dark state with RVB character. We have presented two limiting cases of this scenario: the first is a minimal RVB state with one resonating dimer, while the second has maximal RVB nature with  $N/2$  dimers. In fact, RVB character is a general property of dark states in this setup. For instance, in the case of half the spins in the excited state initially, if only one photon is emitted over many loss cycles, the final spin state collapses onto  $|S_{tot} = 1, m_{tot} = -1\rangle$ . This is a dark state (in the sense that it cannot emit photons). In fact, it has RVB character and is analogous to a doped Mott insulator. Such RVB states can also be realized from initial states with an unequal number of up and down spins[25]. Details will be presented elsewhere.

The RVB state shown in Fig. 2 is doubly dark as it can neither absorb nor emit photons. This is a key feature that distinguishes it from other dark states such as the aforementioned  $|S_{tot} = 1, m_{tot} = -1\rangle$  state which can absorb but not emit. This suggests a clear experimental signature as the spins completely decouple from the photon mode. When external photons are pumped in, the cavity's transmission characteristics will be the same as that for an empty cavity without spins. This signature can be used to detect false positives arising from inefficiencies in the photon detectors.

One of the limiting factors in our proposal may be the inter-qubit coupling. Interactions, e.g. of dipolar character, may cause dephasing. However, interactions may be useful for creating more interesting RVB states,

by inducing a preference for shorter range dimers. This is a very interesting direction as short range RVB states are known to harbour topological order.

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### Applying Burnside's lemma to the permutation group

With the objective of proving Eq. 11 of the main text, we now apply Burnside's lemma to the permutation group.

Consider a finite group  $G$  that acts on a finite set  $X$ , i.e.,  $Gx \in X$ , for every  $x \in X$ . Burnside's lemma states that the number of orbits for the action of  $G$  on  $X$  is given by

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|, \quad (17)$$

where  $X^g = \{x \in X | g.x = x\}$ , i.e., the set of elements that are left invariant under the action of  $g$ .

We take  $G$  to be the group of permutations of  $D$  objects. It can be easily seen that permutations satisfy all properties required of a group. Each element of this group corresponds to one allowed dimer cover, as explained in the main text. The number of elements of  $G$  is given by  $|G| = D!$ .

We take the set  $X$  to be the set of configurations of  $D$  objects in which each object may have one of two attributes, e.g., the set of  $D$  balls with each ball being red (R) or blue (B). The total number of elements in  $X$  is  $2^D$ , since each ball may have one of two colours. Let us now identify configurations that are connected by a permutation. The set  $X$  is now partitioned into 'orbits'; configurations within each orbit are related to one another by a permutation operation. For instance, for  $D = 4$ , the configurations  $\{RRBB, RBBR, BBRR, BRRB, RBRB, BRBR\}$  form an orbit. To find the number of orbits, denoted by  $|X/G|$ , we note that each orbit is completely characterized by the number of R's in the configuration. As the number of R's can range from 0 to  $D$ , we have  $(D + 1)$  independent orbits.

We now determine the quantity  $|X^g|$  occurring in Eq. 17. This is the number of configurations that are left invariant by one particular element of  $G$  (one particular permutation of the  $D$  balls). This can be deduced as follows: for each permutation, we have  $N_l(P)$  number of 'cycles', i.e., the  $D$  objects are partitioned into

$N_l(P)$  subsets such that the permutation operation  $P$  only displaces objects within each subset. The necessary condition for a configuration to be invariant under  $P$  is that each cycle will have elements of the same colour. In other words, we may independently assign a colour to each cycle. Thus, the quantity  $|X^g|$  is given by  $2^{N_l(P)}$ . We note that the number of cycles  $N_l(P)$  is the same as the number of loops discussed in main text.

Eq. 17 now gives us

$$\sum_P 2^{N_l(P)} = D!(D + 1), \quad (18)$$

the result given in Eq. 11 of the main text.

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